

$$1. \int_0^{2\pi} \int_0^{\pi} y^2 \cos(x) \sin(7y) dy dx$$

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$$\begin{array}{lll} \text{Let} & & \\ u = y^2 & du = 2y dy & \\ dv = \sin(7y) & v = -\frac{1}{7} \cos(7y) & = uv - \int v du. \end{array}$$

$$= \cos(x) \left(-\frac{y^2}{7} \cos(7y) + \int \frac{2}{7} y \cos(7y) dy \right)$$

$$\int \frac{2}{7} y \cos(7y) dy = \frac{2}{7} \int y \cos(7y) dy$$

$$\text{Let } u = 7y, du = 7dy, \frac{dy}{7} = dy$$

$$= \frac{2}{7} \int \frac{1}{49} u \cos u du$$

$$= \frac{2}{343} \int u \cos u du \quad \begin{array}{ll} \text{Let } u = u & du = du \\ v' = \cos u & v = \sin u \end{array}$$

$$= \frac{2}{343} \left[-u \sin u + \int \sin u du \right]$$

$$= \frac{2}{343} \left(-u \sin u - \cos u \right) = \frac{2}{343} \left(-7y \sin(7y) - \cos(7y) \right)$$

$$= \cos(x) \left(-\frac{y^2}{7} \cos(7y) + \frac{2}{343} \left(-7y \sin(7y) - \cos(7y) \right) \right) \Big|_0^{\pi}$$

$$= \left(\frac{\pi^2}{7} - \frac{2}{343} \right) \cos(x) - \frac{2}{343} \cos(x) = \frac{\pi^2}{7} \cos(x) - \frac{4}{343} \cos(x)$$

$$= \int_0^{2\pi} \left(\frac{\pi^2}{7} \cos(x) - \frac{4}{343} \cos(x) \right) dx$$

$$\begin{aligned}
&= \int_0^{2\pi} \frac{\pi^2}{7} \cos(x) dx - \int_0^{2\pi} \frac{4}{343} \cos(x) dx \\
&= \frac{\pi^2}{7} \sin(x) \Big|_0^{2\pi} - \frac{4}{343} \sin(x) \Big|_0^{2\pi} \\
&= \left(\frac{\pi^2}{7} \sin(2\pi) - \frac{\pi^2}{7} \sin(0) \right) - \left(\frac{4}{343} \sin(2\pi) - \frac{4}{343} \sin(0) \right) \\
&= 0
\end{aligned}$$

$$2. \int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1} dy dx$$

$$x=0$$

$$x=1$$

$$y=1$$

$$y=\sqrt{x} \Rightarrow x=y^2$$

$$0 \leq y \leq 1$$

$$0 \leq x \leq y^2$$

$$= \int_0^1 \int_0^{y^2} \sqrt{y^3+1} dx dy$$

$$= \int_0^{y^2} \sqrt{y^3+1} dx$$

$$= \int_0^{y^2} \sqrt{y^3+1} dx = x \sqrt{y^3+1} \Big|_0^{y^2}$$

$$= y^2 \sqrt{y^3+1} - 0 = y^2 \sqrt{y^3+1}$$

$$= \int_0^1 y^2 \sqrt{y^3+1} dy$$

$$\text{Let } u = y^3+1, du = 3y^2 dy, \frac{du}{3} = y^2 dy$$

$$\begin{aligned}
&= \int_0^1 \frac{1}{3} u^{\frac{1}{2}} dy = \frac{1}{3} \left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) \Big|_0^1 = \frac{2}{9} (y^3+1)^{\frac{3}{2}} \Big|_0^1 \\
&= \frac{2}{9} (1+1)^{\frac{3}{2}} - \frac{2}{9} (0+1)^{\frac{3}{2}}
\end{aligned}$$

$$= \frac{4\sqrt{2}}{9} - \frac{2}{9}$$

$$3 \iint_D (x^3 + y^3 + \sqrt{9-x^2}) dA \text{ where } D = [-3, 3] \times [-1, 1]$$

$$= \int_{-3}^3 \int_{-1}^1 (x^3 + y^3 + \sqrt{9-x^2}) dy dx$$

$$= \int_{-3}^3 (x^3 + y^3 + \sqrt{9-x^2}) dy$$

$$= x^3 y + \frac{y^4}{4} + y \sqrt{9-x^2} \Big|_{-1}^1$$

$$= \left(x^3 + \frac{1}{4} + \sqrt{9-x^2} \right) - \left(-x^3 + \frac{1}{4} - \sqrt{9-x^2} \right)$$

$$= 2x^3 + 2\sqrt{9-x^2}$$

$$= \int_{-3}^3 (2x^3 + 2\sqrt{9-x^2}) dx$$

$$= \int_{-3}^3 2x^3 dx + \int_{-3}^3 2\sqrt{9-x^2} dx$$

$$\int_{-3}^3 2\sqrt{9-x^2} dx = 2 \int_{-3}^3 \sqrt{3^2-x^2} dx$$

$$\text{Let: } x = 3\sin(u), dx = 3\cos u du.$$

$$= 2 \int_{-3}^3 9 \cos^2 u du = 18 \int_{-3}^3 \frac{1 + \cos(2u)}{2} du.$$

$$= 9 \int_{-3}^3 1 + \cos(2u) du$$

$$= 9u + \frac{9}{2} \sin(2u)$$

$$= 9\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2} \sin\left(2\sin^{-1}\left(\frac{x}{3}\right)\right) \Big|_{-3}^3$$

$$= \int_{-3}^3 2x^3 dx = \frac{x^4}{2} \Big|_{-3}^3$$

$$= \left(\frac{(3)^4}{2} - \frac{(-3)^4}{2} \right) - \left(9 \sin^{-1}\left(\frac{3}{3}\right) + \frac{9}{2} \sin(2 \sin^{-1}\left(\frac{3}{3}\right)) - 9 \sin^{-1}\left(-\frac{3}{3}\right) - \frac{9}{2} \sin(2 \sin^{-1}\left(-\frac{3}{3}\right)) \right)$$

$$= \frac{9\pi + 81}{2} - \frac{-9\pi + 81}{2} = 9\pi$$

$$5. \int_0^1 \int_0^{2-2x} \int_0^{8-8x-4y} 3 dz dy dx.$$

$$= \int_0^1 \int_0^{2-2x} 3 dz = 3z \Big|_0^{8-8x-4y} = 3(8-8x-4y)$$

$$= \int_0^1 3(8-8x-4y) dy = 3(8y-8xy-2y^2) \Big|_0^{2-2x}$$

$$= 3(8(2-2x)-8x(2-2x)-2(2-2x)^2)$$

$$= 3(8x^2-16x+8)$$

$$= \int_0^1 3(8x^2-16x+8) = 3\left(\frac{8x^3}{3}-8x^2+8x\right) \Big|_0^1 = 8-0$$

$$= 8$$

$$7. \int_0^1 \int_0^1 \sqrt{1-y^2} \, dx \, dy$$

$$= \int_0^1 \sqrt{1-y^2} \, dx = x\sqrt{1-y^2} \Big|_0^1 = \sqrt{1-y^2}$$

$$= \int_0^1 \sqrt{1-y^2} \, dy =$$

$$y = \sin u \quad dy = \cos u$$

$$= \int_0^1 \cos^2 u \, du = \int_0^1 \frac{1}{2} (1 + \cos 2u) \, du$$

$$= \frac{1}{2} \int_0^1 (1 + \cos 2u) \, du = \frac{1}{2} \left(u + \frac{1}{2} \sin 2u \right) \Big|_0^1$$

$$= \frac{1}{2} \left(\sin^{-1} y + \frac{1}{2} \sin(\sin^{-1} y) \right) \Big|_0^1$$

$$= \frac{1}{2} \left(\sin^{-1}(1) + \frac{1}{2} \sin(\sin^{-1}(1)) \right) - \frac{1}{2} \left(\sin^{-1}(0) + \frac{1}{2} \sin(\sin^{-1}(0)) \right)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$8. \quad Z = x^2 + y^2 \quad x^2 + y^2 \leq 4$$

$$= \iint_D \sqrt{1 + (Z_x)^2 + (Z_y)^2} \, dA$$

$$Z_x = 2x$$

$$Z_y = 2y$$

$$\sqrt{1 + (2x)^2 + (2y)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$= \iint_D \sqrt{1 + 4x^2 + 4y^2} \, dA$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$x^2 + y^2 = r^2 = 4$$

$$4x^2 + 4y^2 = 4r^2$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{1+4r^2} r dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 r(1+4r^2)^{\frac{1}{2}} dr d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} (1+4r^2)^{\frac{3}{2}} \Big|_0^2 d\theta$$

$$\int_0^{2\pi} \frac{17\sqrt{17}-1}{2} d\theta$$

$$\left(\frac{17\sqrt{17}-1}{12} \theta \right) \Big|_0^{2\pi}$$

$$\frac{2\pi(17\sqrt{17}-1)}{12}$$

$$= \frac{\pi(17\sqrt{17}-1)}{6}$$

$$10. \quad x^2 + y^2 + z^2 = 4, \quad z^2 = x^2 + y^2$$

$$z^2 = 4 - x^2 - y^2$$

$$4 - x^2 - y^2 = x^2 + y^2$$

$$4 = 2x^2 + 2y^2$$

$$2 = x^2 + y^2$$

$$= \int_{-\pi}^{\pi} \int_0^{\sqrt{2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} r \, dz \, dr \, d\theta$$

$$= \int_{-\pi}^{\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$$

$$= \int_{-\pi}^{\pi} \int_0^{\sqrt{2}} r (\sqrt{4-r^2} - r^2) \, dr \, d\theta$$

$$= \int_{-\pi}^{\pi} \frac{-4\sqrt{2} + 8}{3} \, d\theta$$

$$= \left(-\frac{4\sqrt{2}+8}{3} \right) e \bigg|_{-\pi}^{\pi}$$

$$= \frac{2\pi(-4\sqrt{2}+8)}{3}$$

$$11 \iiint_V z^2 dx dy dz \quad \begin{aligned} x^2 + y^2 + z^2 &= 1 \\ z^2 &= 1 - x^2 - y^2 = 1 - r^2 \end{aligned}$$

$$= \int_0^1 \int_0^{2\pi} \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r(1-r^2) dz d\theta dr$$

$$\int_0^1 \int_0^{2\pi} 2r(1-r^2)\sqrt{1-r^2} d\theta dr$$

$$= 2\pi \int_0^1 2r(1-r^2)\sqrt{1-r^2} dr$$

$$\begin{aligned} r &= \sin t, \quad \sqrt{1-r^2} = \cos t \\ dr &= \cos t dt, \quad t(0) = 0, \quad t(1) = \frac{\pi}{2} \end{aligned}$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \sin t \cos^2 t (\cos t) dt$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \sin t \cos^3(t) dt$$

$$= 4\pi \left[-\frac{1}{4} \cos^4 t \right]_0^{\frac{\pi}{2}}$$

$$= 4\pi \left[-\frac{1}{4} \cos^4\left(\frac{\pi}{2}\right) + \frac{1}{4} \cos^4(0) \right]$$

$$= 4\pi (0 + \frac{1}{4})$$

$$= \pi$$

12 $\iint_R xy \, dA$ $xy=1$ $y=x$
 $xy=2$ $y=3x$

$$u = xy \quad v = \frac{y}{x}$$

$$u = xy \quad \frac{y}{x} = x^2 \quad x = u^{\frac{1}{2}} v^{-\frac{1}{2}}$$

$$v = \frac{y}{x} \quad uv = y^2 \quad y = u^{\frac{1}{2}} v^{\frac{1}{2}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} u^{-\frac{1}{2}} v^{-\frac{1}{2}} & -\frac{1}{2} u^{\frac{1}{2}} v^{-\frac{3}{2}} \\ \frac{1}{2} u^{-\frac{1}{2}} v^{\frac{1}{2}} & \frac{1}{2} u^{\frac{1}{2}} v^{-\frac{1}{2}} \end{vmatrix}$$

$$= \frac{1}{4} v^{-1} + \frac{1}{4} v^{-1} = \frac{1}{2v}$$

$$\iint_R xy \, dA = \int_1^3 \int_1^3 u \frac{1}{2v} = \frac{1}{2} \int_1^3 u \, du \int_1^3 \frac{1}{v} \, dv$$

$$= 2 \ln(3)$$

13 $f(x,y) = x+y$ $x = 3e^t + 1$ $y = 4e^t - 1$

$$dx = 3e^t dt, \quad dy = 4e^t dt$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{(3e^t)^2 + (4e^t)^2}$$

$$= \sqrt{9e^{2t} + 16e^{2t}}$$

$$= \sqrt{25e^{2t}} = 5e^t dt$$

$$= \int_0^{\ln 2} (3e^t + 4e^t) 5e^t dt$$

$$= \int_0^{\ln 2} 35e^{2t} dt =$$

$$\frac{35}{2} e^{2t} \Big|_0^{\ln(2)} = \frac{35}{2} [e^{2\ln 2} - e^{2(0)}]$$

$$= \frac{35}{2} [4 - 1]$$

$$= \frac{105}{2}$$

$$14. = \int_0^{2\pi} \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

$$y = r \sin \theta \quad x = r \cos \theta$$

$$dy = r \cos \theta d\theta \quad dx = -r \sin \theta d\theta$$

$$r^2 = x^2 + y^2$$

$$= \int_0^{2\pi} (\sin^2 \theta + \cos^2 \theta) d\theta$$

$$= \int_0^{2\pi} 1 d\theta = \theta \Big|_0^{2\pi}$$

$$= 2\pi$$